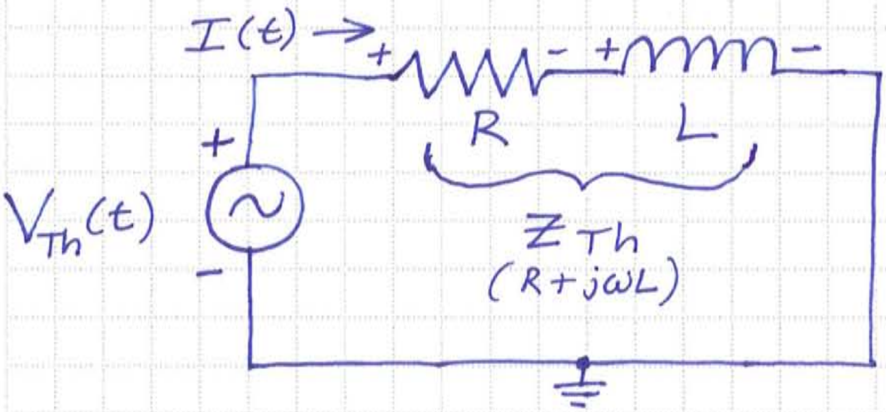
Derive Time Domain Equation for Short Circuit Current in Series RL Circuit (Transient + Steady State Response)

For purpose of fault calculation, network has been reduced to Thevenin equivalent open-circuit voltage and Thevenin equivalent impedance at fault location:



$V_{Th}(t) = V(t)$

$V(t) = V_m \sin(\omega t + \phi)$

$V_m = V_{maximum}$

ω = angular frequency

ϕ = phase

Apply Kirchoff's Voltage Law,

$$\text{KVL:} \quad \sum V_{LOOP} = 0$$

$$-V(t) + RI + L\frac{dI}{dt} = 0$$

$$V(t) = RI(t) + L\frac{dI(t)}{dt}$$

$$V_m \sin(\omega t + \phi) = RI(t) + L\frac{dI(t)}{dt}$$

Linear First-Order Differential Equation: Solve for $I(t)$

Place first-order equation in proper form:

$$L\frac{dI(t)}{dt} + RI(t) = V_m \sin(\omega t + \phi)$$

$$\frac{dI(t)}{dt} + \frac{R}{L}I(t) = \frac{V_m}{L}\sin(\omega t + \phi)$$

Method 1: Solve first-order equation using integrating factor

$$\mu(t) = e^{\int P(t)dt}$$

where equation is of the form $\frac{dy}{dt} + P(t)y = Q(t)$

$$\mu(t) = e^{\int (\frac{R}{L})dt} = e^{(R/L)t}$$

multiply both sides of equation by integrating factor,

$$\underbrace{e^{(R/L)t}\cdot\frac{dI(t)}{dt} + e^{(R/L)t}\cdot\frac{R}{L}I(t)} = e^{(R/L)t}\cdot\frac{V_m}{L}\sin(\omega t + \phi)$$

$$\frac{d}{dt}\left(e^{(R/L)t}\cdot I(t)\right) = e^{(\frac{R}{L})t}\cdot\frac{V_m}{L}\sin(\omega t + \phi)$$

Integrate both sides,

$$e^{(R/L)t}\cdot I(t) = \int e^{(R/L)t}\cdot\frac{V_m}{L}\sin(\omega t + \phi)dt + C$$

$$I(t) = \frac{1}{e^{(R/L)t}}\cdot\left[\int e^{(R/L)t}\cdot\frac{V_m}{L}\sin(\omega t + \phi)dt + C\right]$$

$$I(t) = e^{-(\frac{R}{L})t}\cdot\left[\frac{V_m}{L}\left(\int e^{(R/L)t}\cdot\sin(\omega t + \phi)dt\right) + C\right]$$

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Use trigonometric Identity

$$\sin(x+y) = \cos x \sin y + \cos y \sin x$$

$$I(t) = e^{-(R/L)t} \cdot \left[\frac{V_m}{L} \left(\int e^{(R/L)t} (\cos\omega t \sin\phi + \cos\phi \sin\omega t)\, dt \right) + C \right]$$

$$I(t) = e^{-(R/L)t} \cdot \left[\frac{V_m}{L} \left(\int e^{(R/L)t} \cos\omega t \sin\phi\, dt + \int e^{(R/L)t} \cos\phi \sin\omega t\, dt \right) + C \right]$$

$$I(t) = e^{-(R/L)t} \cdot \left[\frac{V_m}{L} \left(\sin\phi \int e^{(R/L)t} \cos\omega t\, dt + \cos\phi \int e^{(R/L)t} \sin\omega t\, dt \right) + C \right]$$

From Table of Integrals,

$$\int e^{au} \sin bu\, du = \frac{e^{au}}{a^2+b^2}(a \sin bu - b \cos bu) + c$$

$$\int e^{au} \cos bu\, du = \frac{e^{au}}{a^2+b^2}(a \cos bu + b \sin bu) + c$$

So that,

$$I(t) = e^{-(R/L)t} \cdot \left[\frac{V_m}{L} \left(\sin\phi \left(\frac{e^{(R/L)t}}{(R/L)^2+\omega^2} \left(\frac{R}{L} \cos\omega t + \omega \sin\omega t \right) \right) + \cos\phi \left(\frac{e^{(R/L)t}}{(R/L)^2+\omega^2} \left(\frac{R}{L} \sin\omega t - \omega \cos\omega t \right) \right) \right) + C \right]$$

$$I(t) = e^{-(R/L)t} \cdot \left[\frac{V_m}{L} \cdot \frac{e^{(R/L)t}}{(R/L)^2+\omega^2} \left(\frac{R}{L} \cos\omega t \sin\phi + \omega \sin\omega t \sin\phi + \frac{R}{L} \sin\omega t \cos\phi - \omega \cos\omega t \cos\phi \right) + C \right]$$

$$I(t) = e^{-(R/L)t} \cdot \left[\frac{V_m}{L} \cdot \frac{e^{(R/L)t}}{(R/L)^2+\omega^2} \cdot \left(\frac{R}{L} (\cos\omega t \sin\phi + \sin\omega t \cos\phi) - \omega (\cos\omega t \cos\phi - \sin\omega t \sin\phi) \right) + C \right]$$

Use trigonometric Identities

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$I(t) = e^{-(R/L)t} \cdot \left[\frac{V_m}{L} \cdot \frac{e^{(R/L)t}}{(R/L)^2+\omega^2}\left(\frac{R}{L}\sin(\omega t+\phi) - \omega\cos(\omega t+\phi)\right) + C\right]$$

$$I(t) = \frac{V_m}{L} \cdot \frac{1}{(R/L)^2+\omega^2}\left[\frac{R}{L}\sin(\omega t+\phi) - \omega\cos(\omega t+\phi)\right] + Ce^{-(R/L)t}$$

$$I(t) = \frac{V_m}{L} \cdot \frac{L^2}{R^2+(\omega L)^2}\left[\frac{R}{L}\sin(\omega t+\phi) - \omega\cos(\omega t+\phi)\right] + Ce^{-(R/L)t}$$

$$I(t) = \frac{V_m}{R^2+(\omega L)^2}\left[R\sin(\omega t+\phi) - \omega L\cos(\omega t+\phi)\right] + Ce^{-(R/L)t}$$

but $R^2+(\omega L)^2 = R^2+X^2 = |Z|^2$ so that

$$I(t) = \frac{V_m}{|Z|^2}\left[R\sin(\omega t+\phi) - \omega L\cos(\omega t+\phi)\right] + Ce^{-(R/L)t}$$

Assume that at $t=0$, $I(0)=0$ so that,

$$0 = \frac{V_m}{|Z|^2}(R\sin\phi - \omega L\cos\phi) + C$$

$$\Rightarrow C = -\frac{V_m}{|Z|^2}(R\sin\phi - \omega L\cos\phi), \text{ then}$$

$$I(t) = \frac{V_m}{|Z|^2}\left[R\sin(\omega t+\phi) - \omega L\cos(\omega t+\phi)\right] - \frac{V_m}{|Z|^2}(R\sin\phi - \omega L\cos\phi)e^{-(R/L)t}$$

$$I(t) = \frac{V_m}{|Z|^2}\left[R\sin(\omega t+\phi) - \omega L\cos(\omega t+\phi) - e^{-(R/L)t}(R\sin\phi - \omega L\cos\phi)\right]$$

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Use trigonometric Identity

$$a\sin x - b\cos x = \pm\sqrt{a^2+b^2}\,\sin\left(x - \tan^{-1}\frac{b}{a}\right)$$

derived from $\sin(x-\theta) = \sin x\cos\theta - \cos x\sin\theta$ as follows:

$a\sin x - b\cos x = c$

Let $a = d\cos\theta$

$b = d\sin\theta$

$c = d\sin(x-\theta)$

$a^2+b^2 = d^2\cos^2\theta + d^2\sin^2\theta$

$d = \pm\sqrt{a^2+b^2}$

$\dfrac{d\sin\theta}{d\cos\theta} = \dfrac{b}{a} = \tan\theta$

$\theta = \tan^{-1}\dfrac{b}{a}$

$\sin(x-\theta) = \sin x\cos\theta - \cos x\sin\theta$

$\Downarrow$

$\underbrace{d\sin(x-\theta)}_{c} = \underbrace{d\cos\theta}_{a}\sin x - \underbrace{d\sin\theta}_{b}\cos x$

$c = a\sin x - b\cos x$

$\Downarrow$

$a\sin x - b\cos x = \pm\sqrt{a^2+b^2}\,\sin(x-\theta)$

Case 1 (first term):

$R\sin(\omega t+\phi) - \omega L\cos(\omega t+\phi)$

$a\sin x - b\cos x$ $\quad x = \omega t+\phi$

$\Rightarrow \pm\sqrt{R^2+(\omega L)^2}\,\sin(\omega t+\phi-\theta)$ $\quad \theta = \tan^{-1}\dfrac{\omega L}{R} = \tan^{-1}\dfrac{X}{R}$

Case 2 (second term):

$R\sin\phi - \omega L\cos\phi$

$a\sin x - b\cos x$ $\quad x = \phi$

$\Rightarrow \pm\sqrt{R^2+(\omega L)^2}\,\sin(\phi-\theta)$ $\quad \theta = \tan^{-1}\dfrac{\omega L}{R} = \tan^{-1}\dfrac{X}{R}$

Then,

$$I(t) = \frac{V_m}{|Z|^2}\left[\pm\sqrt{R^2+(\omega L)^2}\sin(\omega t+\phi-\theta) - e^{-(R/L)t}\cdot\pm\sqrt{R^2+(\omega L)^2}\sin(\phi-\theta)\right]$$

but $|Z| = \sqrt{R^2+(\omega L)^2}$ so that,

$$I(t) = \frac{V_m\cdot|Z|}{|Z|^2}\left[\sin(\omega t+\phi-\theta) - e^{-(R/L)t}\sin(\phi-\theta)\right]$$

Finally,

$$I(t) = \frac{V_m}{|Z|}\left[\sin(\omega t+\phi-\theta) - e^{-(R/L)t}\sin(\phi-\theta)\right]$$

noting that,

ω = angular frequency (radians)

ϕ = phase angle (radians), also called closing angle

θ = angle between voltage and current (radians) ($\cos\theta$ = power factor) (system impedance angle)

$$\theta = \tan^{-1}\frac{\omega L}{R} = \tan^{-1}\frac{X}{R}$$

$$I(t) = \underbrace{\frac{V_m}{|Z|}\sin(\omega t+\phi-\theta)}_{\text{AC component steady state}} - \underbrace{\frac{V_m}{|Z|}e^{-(R/L)t}\cdot\sin(\phi-\theta)}_{\text{Decaying DC component Transient}}$$

$$I_{max} = \frac{V_m}{|Z|} = \frac{\sqrt{2}\,V_{rms}}{|Z|} = \sqrt{2}\,I_{rms}$$

(each term)

Method 2: Solve first-order equation Using Laplace Transforms

$$\frac{dI(t)}{dt} + \frac{R}{L} I(t) = \frac{V_m}{L} \sin(\omega t + \phi)$$

From properties of Laplace Transform,

$$\mathcal{L}\left\{\frac{dI(t)}{dt}\right\}(s) = s\,I(s) - f(0^-)$$

$$\mathcal{L}\{I(t)\}(s) = I(s)$$

From Table of Laplace Transform Pairs,

$$\mathcal{L}\{\sin(\omega t + \phi)\}(s) = \frac{s \sin\phi + \omega \cos\phi}{s^2 + \omega^2}$$

Then,

$$s\,I(s) - f(0^-) + \frac{R}{L} I(s) = \frac{V_m}{L} \cdot \frac{s \sin\phi + \omega \cos\phi}{s^2 + \omega^2}$$

Assume that at $t = 0$, $I(0) = 0$ and $f(0^-) = 0$

$$s\,I(s) + \frac{R}{L} I(s) = \frac{V_m}{L} \cdot \frac{s \sin\phi + \omega \cos\phi}{s^2 + \omega^2}$$

$$I(s)\left[s + \frac{R}{L}\right] = \frac{V_m}{L} \cdot \frac{s \sin\phi + \omega \cos\phi}{s^2 + \omega^2}$$

$$I(s) = \frac{1}{s + \frac{R}{L}} \cdot \frac{V_m}{L} \cdot \frac{s \sin\phi + \omega \cos\phi}{s^2 + \omega^2}$$

$$I(s) = \frac{V_m}{L}\left[\frac{s\sin\phi + \omega\cos\phi}{(s+\frac{R}{L})(s^2+\omega^2)}\right]$$

In order to obtain Inverse Laplace Transforms, $\mathcal{L}^{-1}$, need to put equation in proper form using Partial Fraction Decomposition and the Method of Undetermined Coefficients:

Step 1: $I(s) = \frac{P(s)}{Q(s)}$ degree of numerator is less than denominator

∴ proper form, no long division required.

Step 2: Expand into sum of linear and quadratic factors with undetermined coefficients

$$I(s) = \frac{V_m}{L}\left[\frac{As + B\omega}{s^2+\omega^2} + \frac{C}{s+\frac{R}{L}}\right]$$

Step 3: Determine coefficients. Set $\frac{P(s)}{Q(s)}$ equal to factored terms and multiply both sides by denominator

$$\frac{s\sin\phi + \omega\cos\phi}{(s+\frac{R}{L})(s^2+\omega^2)} = \frac{As + B\omega}{s^2+\omega^2} + \frac{C}{s+\frac{R}{L}}$$

$$s\sin\phi + \omega\cos\phi = (As + B\omega)(s+\tfrac{R}{L}) + C(s^2+\omega^2)$$

$$s\sin\phi + \omega\cos\phi = As^2 + A\tfrac{R}{L}s + B\omega s + B\omega\tfrac{R}{L} + Cs^2 + C\omega^2$$

Step 4: Equate powers of s to obtain 3 equations with 3 unknowns

$$s^0: \quad \omega\cos\phi = B\omega\frac{R}{L} + C\omega^2$$

$$s^1: \quad \sin\phi = A\frac{R}{L} + B\omega$$

$$s^2: \quad 0 = A + C$$

Step 5: Solve for coefficients using substitution

$$A: \quad A = -C$$

$$B: \quad \sin\phi = -C\frac{R}{L} + B\omega$$

$$B\omega = \sin\phi + C\frac{R}{L}$$

$$B = \frac{\sin\phi}{\omega} + \frac{C\frac{R}{L}}{\omega}$$

$$C: \quad \omega\cos\phi = \frac{\cancel{\omega}\frac{R}{L}\left(\sin\phi + C\frac{R}{L}\right)}{\cancel{\omega}} + C\omega^2$$

$$\omega\cos\phi = \frac{R}{L}\sin\phi + C\frac{R^2}{L^2} + C\omega^2$$

$$C\left(\frac{R^2}{L^2} + \omega^2\right) = \omega\cos\phi - \frac{R}{L}\sin\phi$$

$$C = \frac{\omega\cos\phi - \frac{R}{L}\sin\phi}{\frac{R^2}{L^2} + \omega^2} = \frac{\omega L^2\cos\phi - RL\sin\phi}{R^2 + (\omega L)^2}$$

$$B = \frac{\sin\phi}{\omega} + \frac{R\omega L\cos\phi - R^2\sin\phi}{\omega(R^2 + (\omega L)^2)}$$

$$A = \frac{RL\sin\phi - \omega L^2\cos\phi}{R^2 + (\omega L)^2}$$

Then,

$$I(s) = \frac{V_m}{L}\left[\frac{RL\sin\phi - \omega L^2\cos\phi}{R^2+(\omega L)^2}\cdot\frac{s}{s^2+\omega^2} + \left(\frac{\sin\phi}{\omega} + \frac{R\omega L\cos\phi - R^2\sin\phi}{\omega(R^2+(\omega L)^2)}\right)\frac{\omega}{s^2+\omega^2}\right.$$

$$\left. + \frac{\omega L^2\cos\phi - RL\sin\phi}{R^2+(\omega L)^2}\cdot\frac{1}{s+\frac{R}{L}}\right]$$

From Table of Laplace Transform Pairs,

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+\omega^2}\right\}(t) = \cos\omega t$$

$$\mathcal{L}^{-1}\left\{\frac{\omega}{s^2+\omega^2}\right\}(t) = \sin\omega t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s+R/L}\right\}(t) = e^{-(R/L)t}$$

so that,

$$\mathcal{L}^{-1}\{I(s)\}(t) = I(t) = \frac{V_m}{L}\left[\cos\omega t\left(\frac{RL\sin\phi - \omega L^2\cos\phi}{R^2+(\omega L)^2}\right)\right.$$

$$+ \sin\omega t\left(\frac{\sin\phi}{\omega} + \frac{R\omega L\cos\phi - R^2\sin\phi}{\omega(R^2+(\omega L)^2)}\right)$$

$$\left. + e^{-(R/L)t}\cdot\left(\frac{\omega L^2\cos\phi - RL\sin\phi}{R^2+(\omega L)^2}\right)\right]$$

$$I(t) = \frac{V_m}{L}\left[\frac{RL\cos\omega t\sin\phi - \omega L^2\cos\omega t\cos\phi}{R^2+(\omega L)^2} + \frac{\sin\omega t\sin\phi}{\omega}\right.$$

$$\left. + \frac{R\omega L\sin\omega t\cos\phi - R^2\sin\omega t\sin\phi}{\omega(R^2+(\omega L)^2)} + e^{-(\frac{R}{L})t}\cdot\frac{\omega L^2\cos\phi - RL\sin\phi}{R^2+(\omega L)^2}\right]$$

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$$I(t) = \frac{V_m}{L}\left[\frac{\omega(RL\cos\omega t\sin\phi - \omega L^2\cos\omega t\cos\phi + (R^2+\omega^2L^2)(\sin\omega t\sin\phi)}{\omega(R^2+\omega^2L^2)} + \frac{R\omega L\sin\omega t\cos\phi - R^2\sin\omega t\sin\phi + e^{-(R/L)t}\cdot(\omega^2L^2\cos\phi - R\omega L\sin\phi)}{\omega(R^2+\omega^2L^2)}\right]$$

$$I(t) = \frac{V_m}{L}\left[\frac{\omega RL\cos\omega t\sin\phi - \omega^2L^2\cos\omega t\cos\phi + \cancel{R^2\sin\omega t\sin\phi}}{\omega(R^2+\omega^2L^2)} + \frac{\omega^2L^2\sin\omega t\sin\phi + R\omega L\sin\omega t\cos\phi - \cancel{R^2\sin\omega t\sin\phi}}{\omega(R^2+\omega^2L^2)} + \frac{e^{-(R/L)t}\cdot(\omega^2L^2\cos\phi - R\omega L\sin\phi)}{\omega(R^2+\omega^2L^2)}\right]$$

$$I(t) = \frac{V_m}{L}\left[\frac{R\omega L(\sin\omega t\cos\phi + \cos\omega t\sin\phi) + \omega^2L^2(\sin\omega t\sin\phi - \cos\omega t\cos\phi)}{\omega(R^2+\omega^2L^2)} + \frac{e^{-(R/L)t}\cdot(\omega^2L^2\cos\phi - R\omega L\sin\phi}{\omega(R^2+\omega^2L^2)}\right]$$

Using trigonometric Identities

$$\sin(x+y) = \sin x\cos y + \cos x\sin y$$
$$\cos(x+y) = \cos x\cos y - \sin x\sin y = -(\sin x\sin y - \cos x\cos y)$$

$$I(t) = \frac{V_m}{L}\left[\frac{R\omega L(\sin(\omega t+\phi)) - \omega^2L^2(\cos(\omega t+\phi)) + e^{-(R/L)t}(\omega^2L^2\cos\phi - R\omega L\sin\phi)}{\omega(R^2+\omega^2L^2)}\right]$$

$$I(t) = V_m\left[\frac{R\omega\sin(\omega t+\phi - \omega^2L\cos(\omega t+\phi) - e^{-(R/L)t}(R\omega\sin\phi - \omega^2L\cos\phi)}{\omega(R^2+\omega^2L^2)}\right]$$

$$I(t) = V_m \left[\frac{R\sin(\omega t+\phi) - \omega L\cos(\omega t+\phi) - e^{-(R/L)t}(R\sin\phi - \omega L\cos\phi)}{R^2 + (\omega L)^2} \right]$$

Using trigonometric Identity

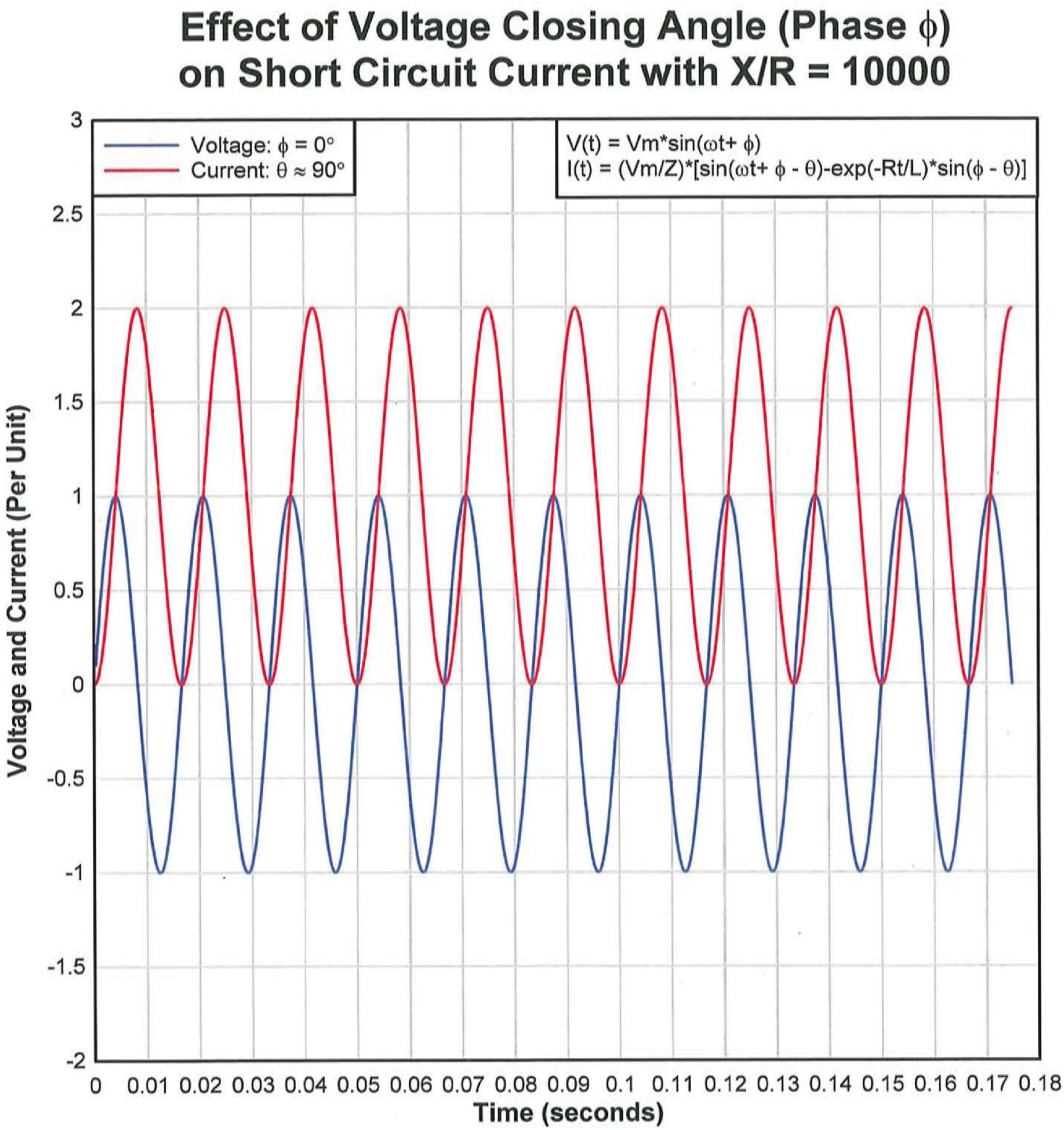
$$a\sin x - b\cos x = \pm\sqrt{a^2+b^2}\,\sin\left(x - \tan^{-1}\frac{b}{a}\right), \quad \theta = \tan^{-1}\frac{b}{a}$$

$$I(t) = \frac{V_m}{R^2+(\omega L)^2}\left[\pm\sqrt{R^2+\omega^2L^2}\,\sin(\omega t+\phi-\theta) - e^{-(R/L)t} \pm\sqrt{R^2+\omega^2L^2}\,\sin(\phi-\theta)\right]$$

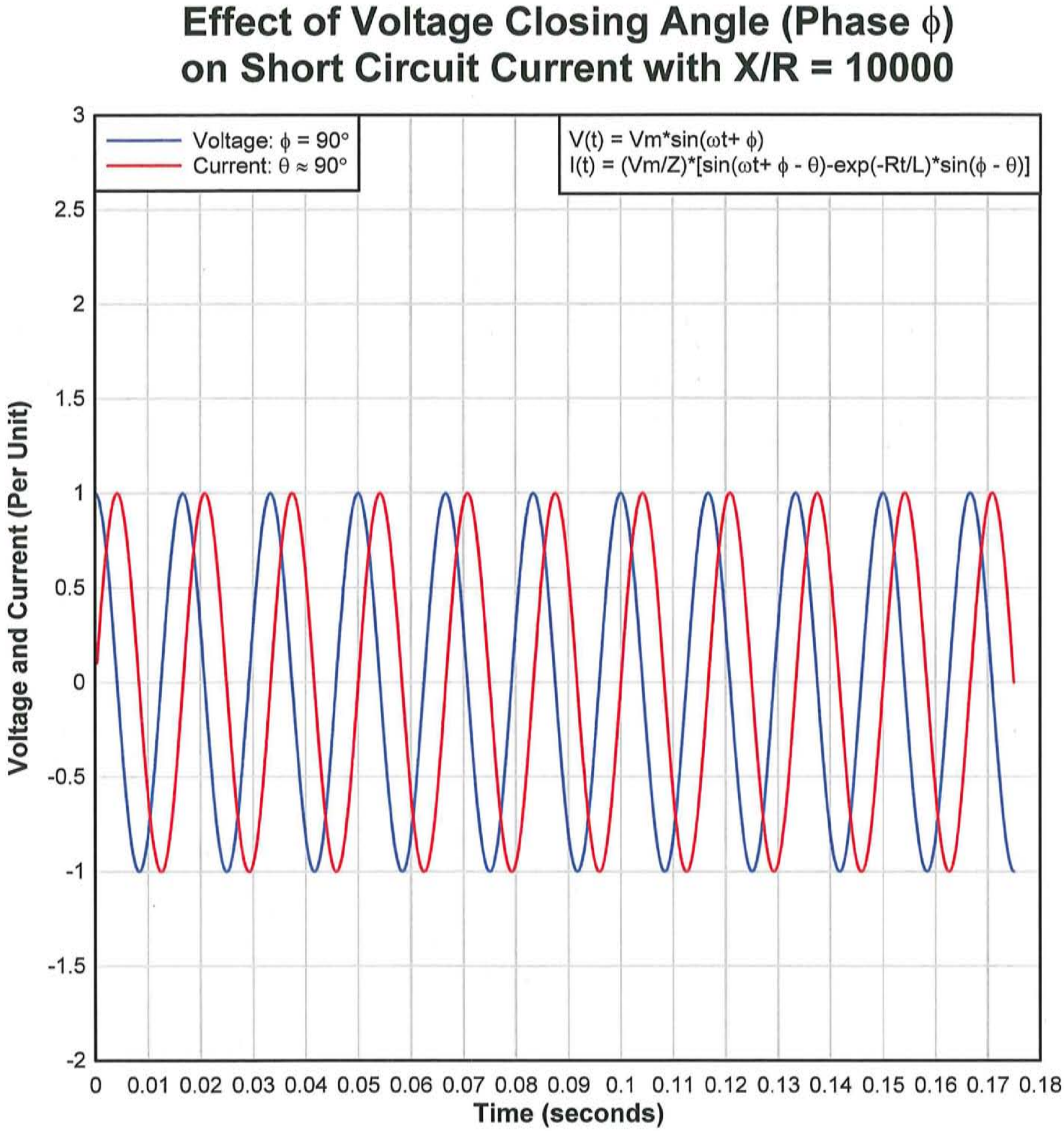
$$|Z| = \sqrt{R^2+\omega^2L^2} \qquad |Z|^2 = R^2+\omega^2L^2$$

$$I(t) = \frac{V_m\,|Z|}{|Z|^2}\left[\sin(\omega t+\phi-\theta) - e^{-(R/L)t}\cdot\sin(\phi-\theta)\right]$$

$$I(t) = \frac{V_m}{|Z|}\left[\sin(\omega t+\phi-\theta) - e^{-(R/L)t}\cdot\sin(\phi-\theta)\right] \checkmark$$

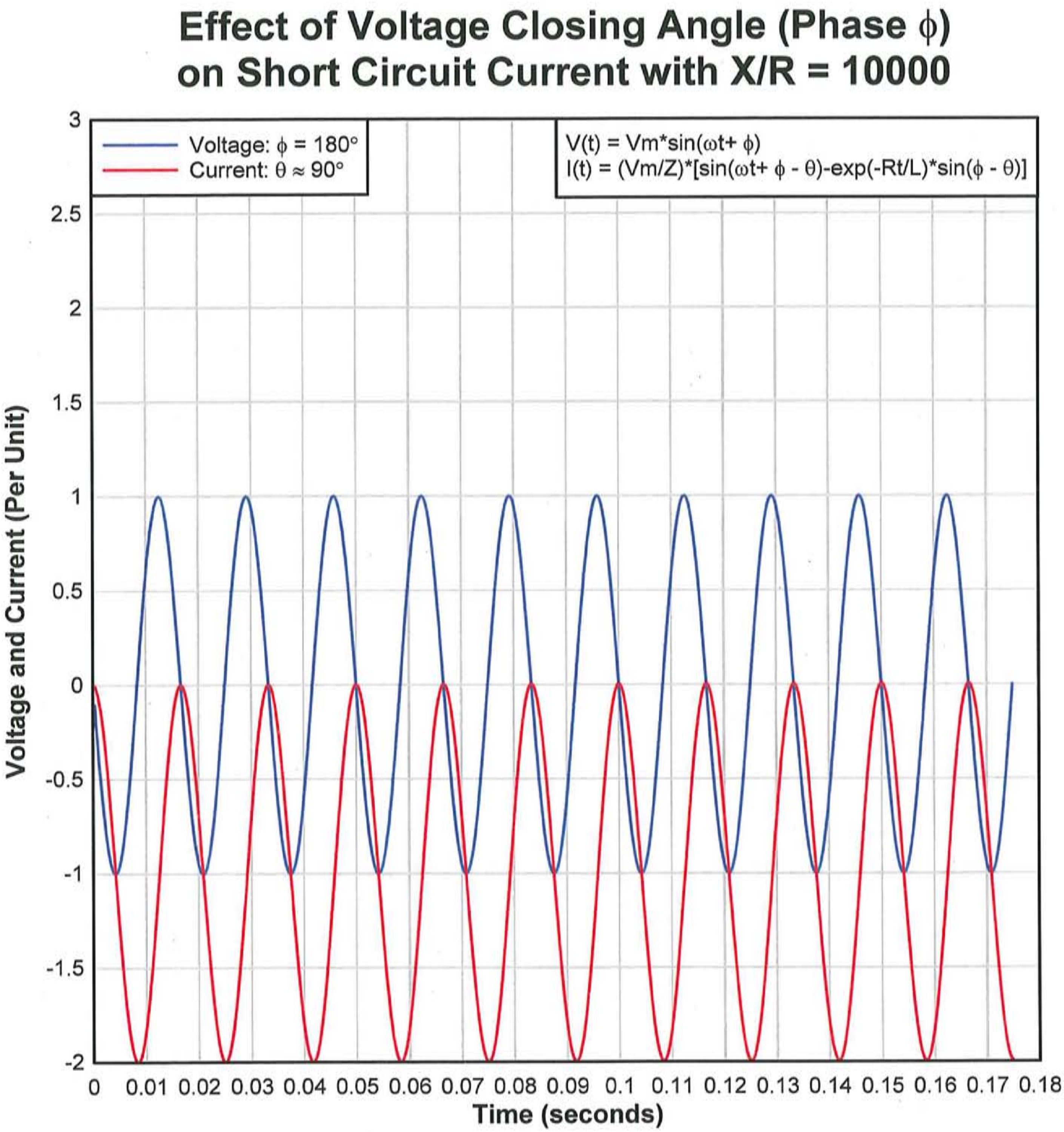


Notes: Produces maximum positive DC offset.



Notes:

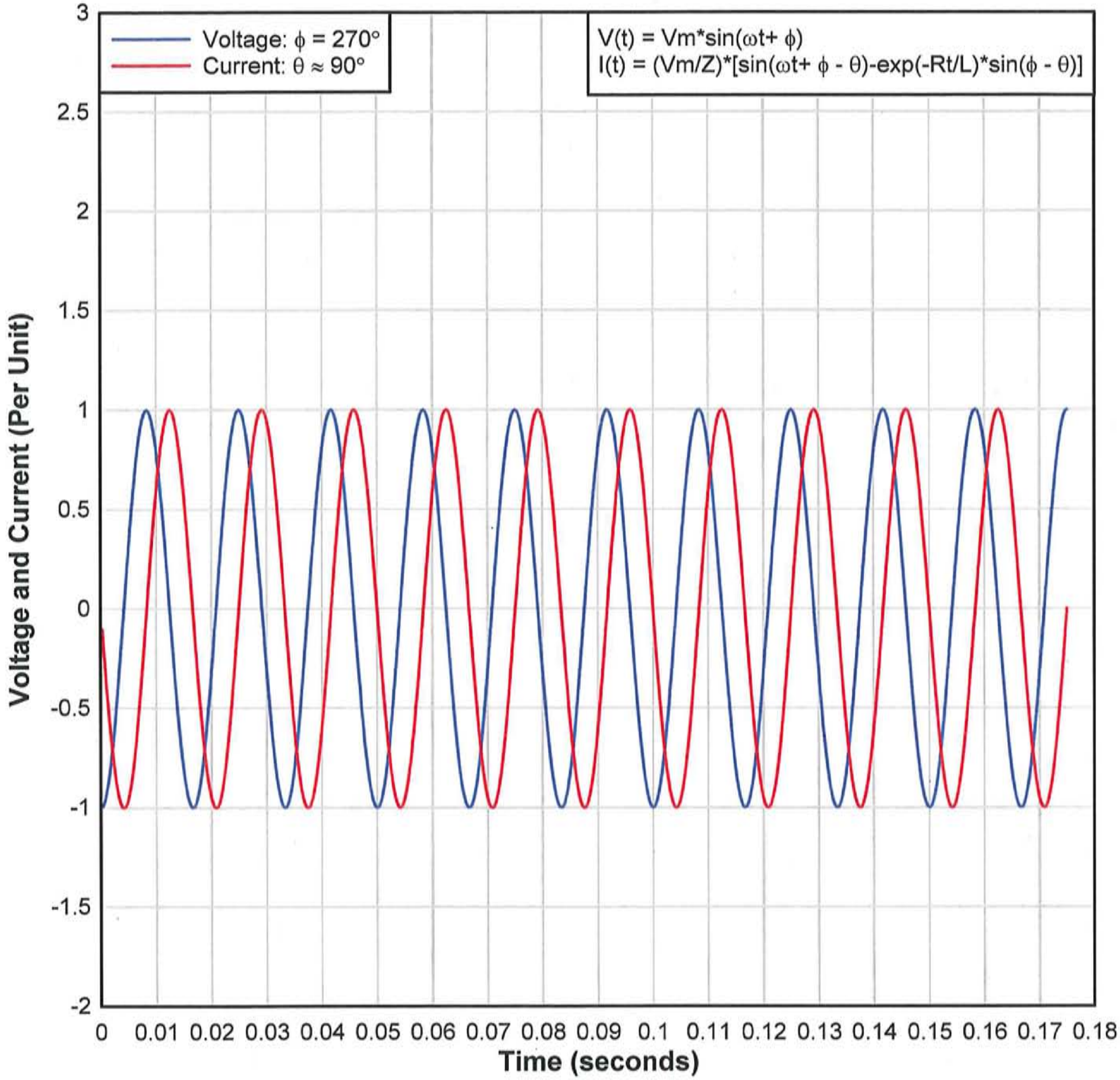
No DC offset



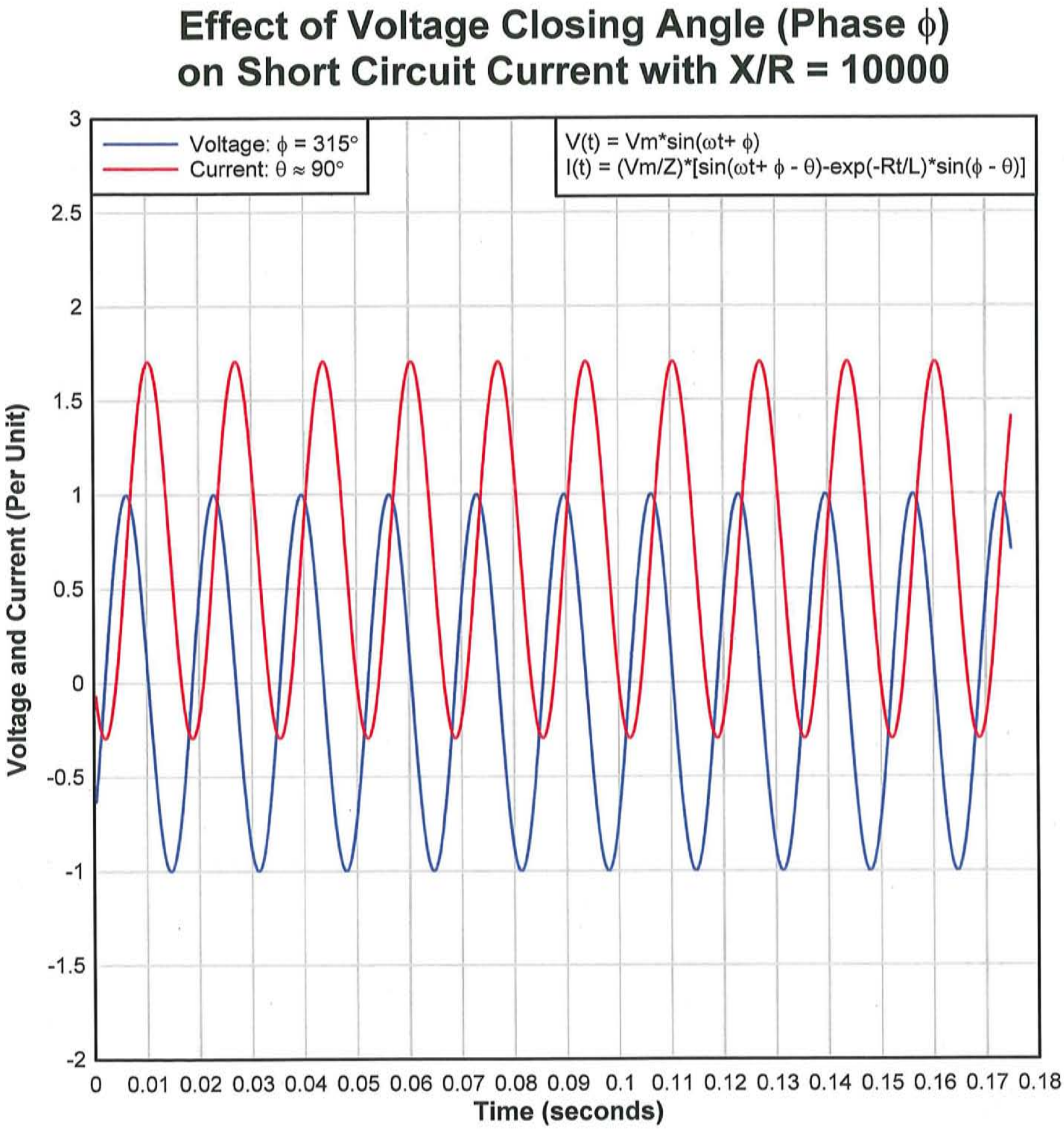
Notes:

Produces maximum negative DC offset.

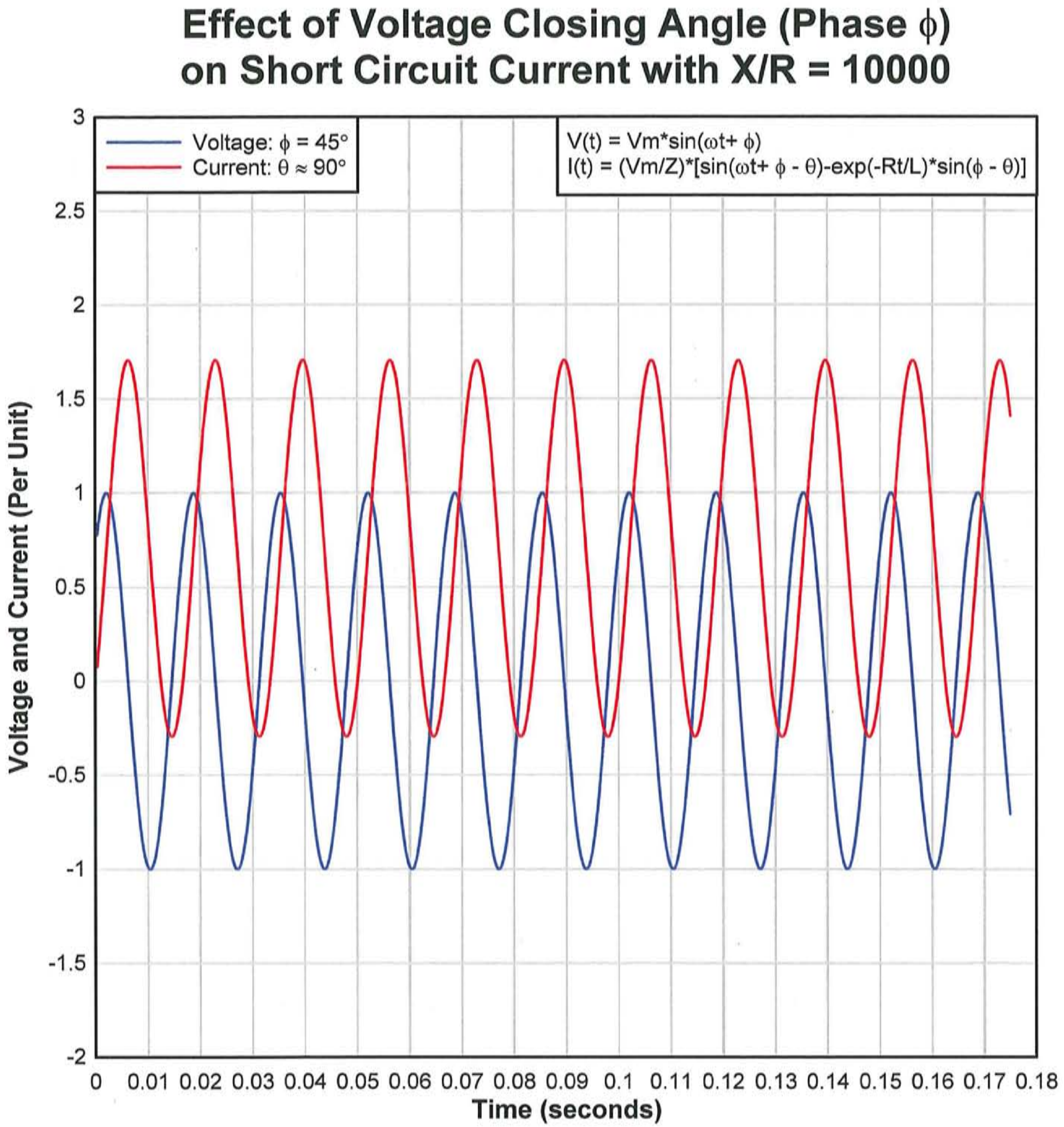
Effect of Voltage Closing Angle (Phase ϕ) on Short Circuit Current with X/R = 10000



Notes: No DC offset.

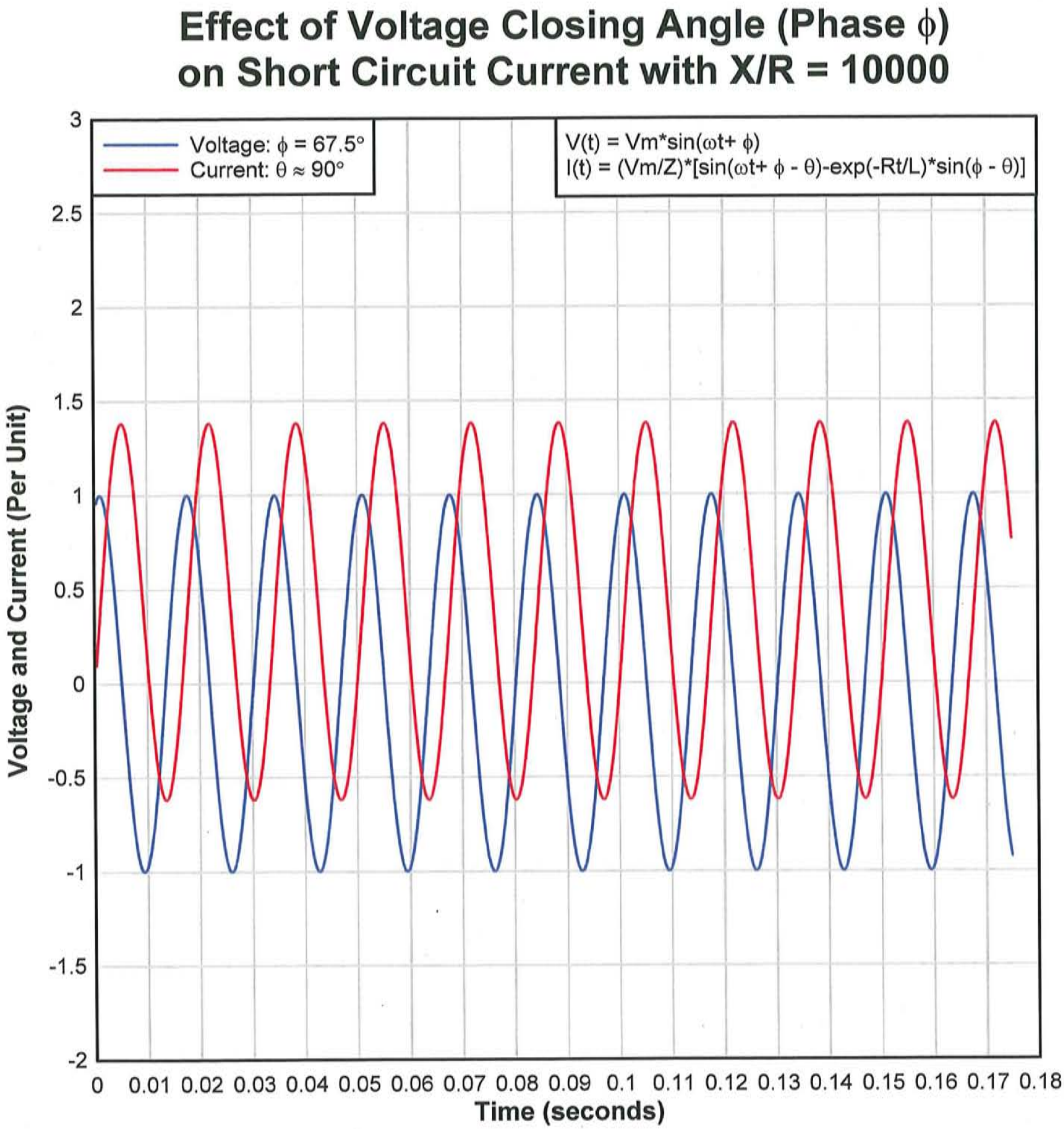


Notes: Intermediate positive DC offset.



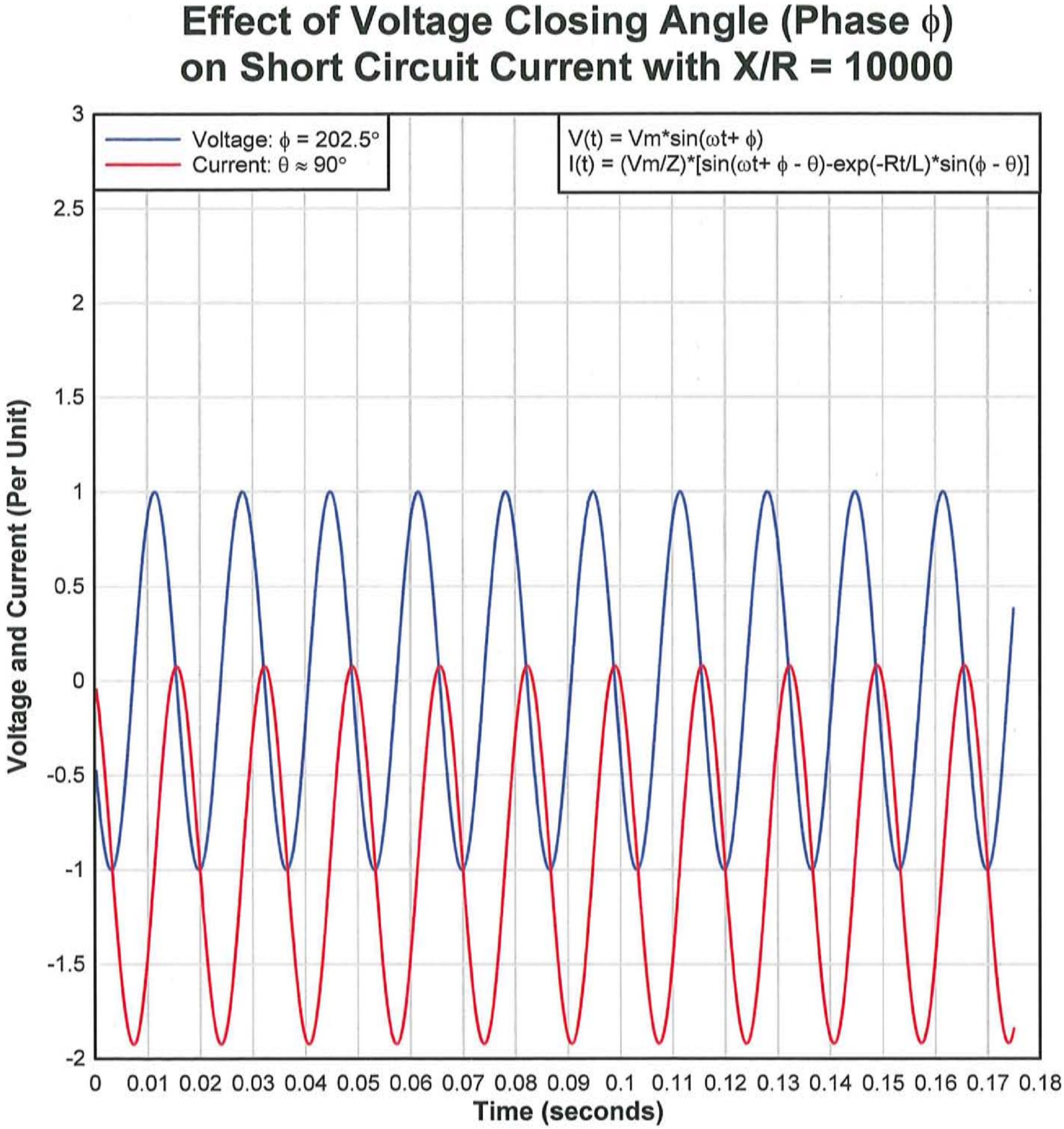
Notes:

Intermediate positive DC offset.



Notes:

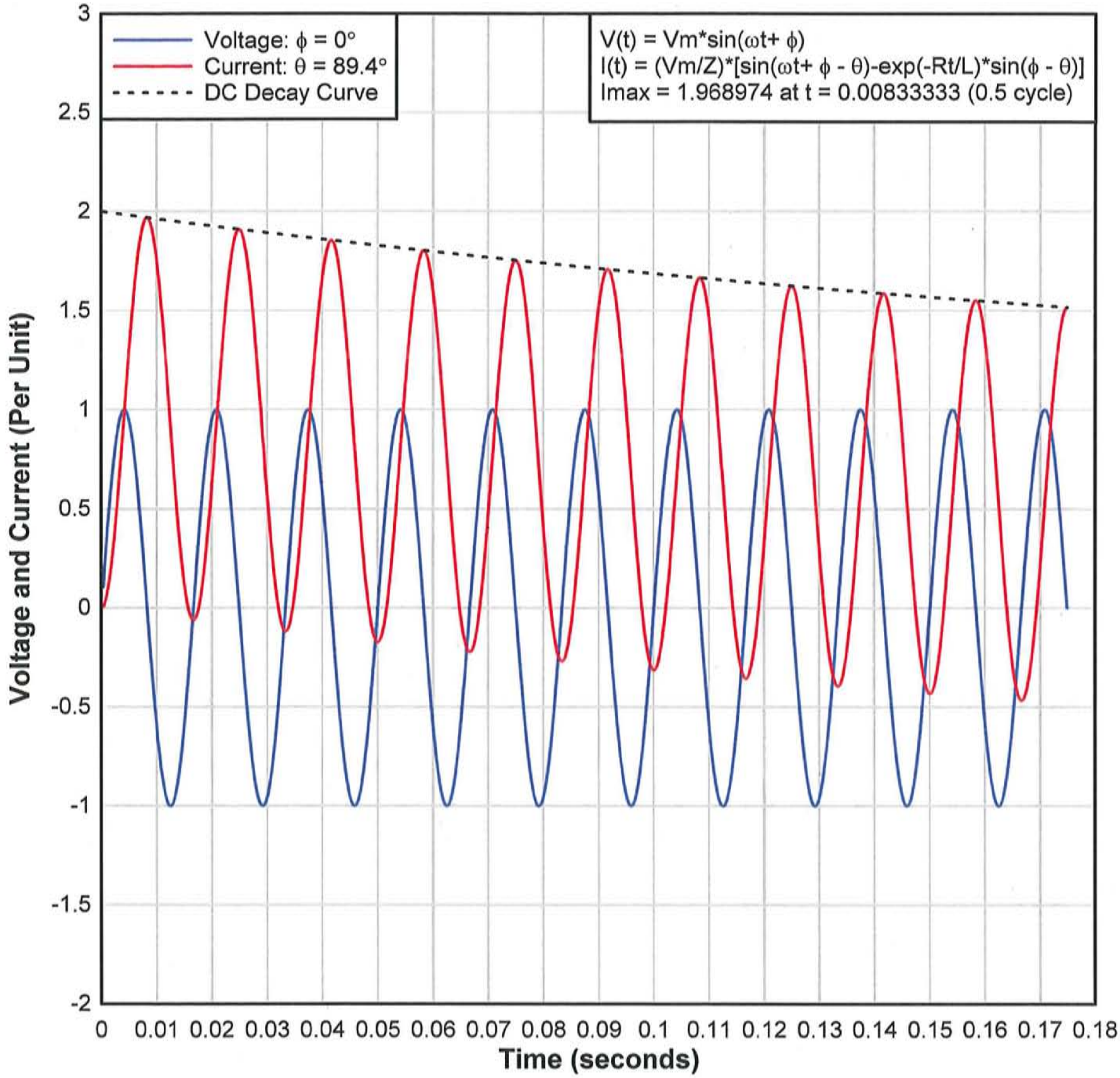
Intermediate positive DC offset.



Notes:

Intermediate negative DC offset.

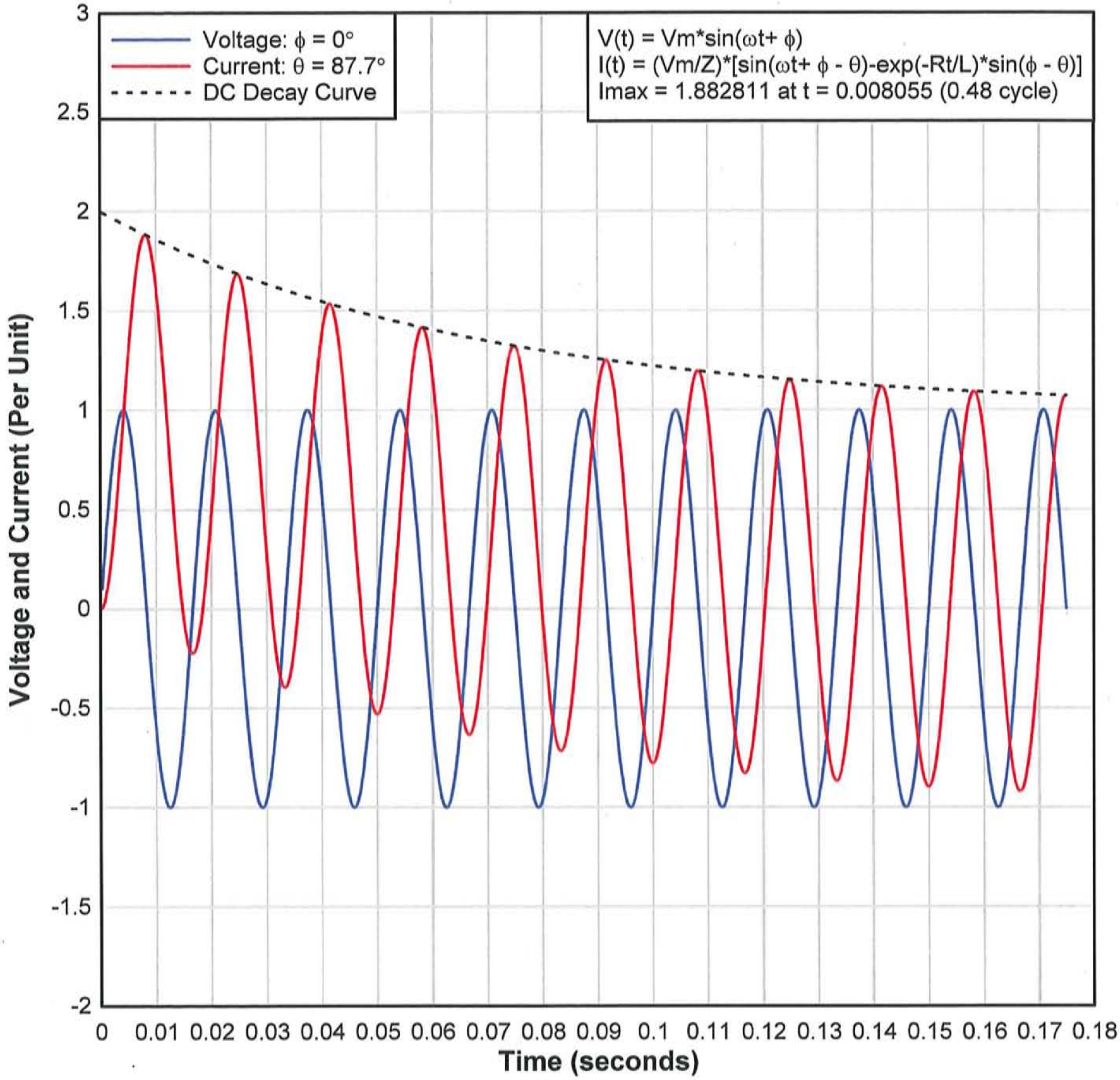
Effect of X/R Ratio on Short Circuit Current
Voltage Closing Angle (Phase ϕ) = 0°
X/R = 100, PF = .001, θ = 89.4°



Notes:

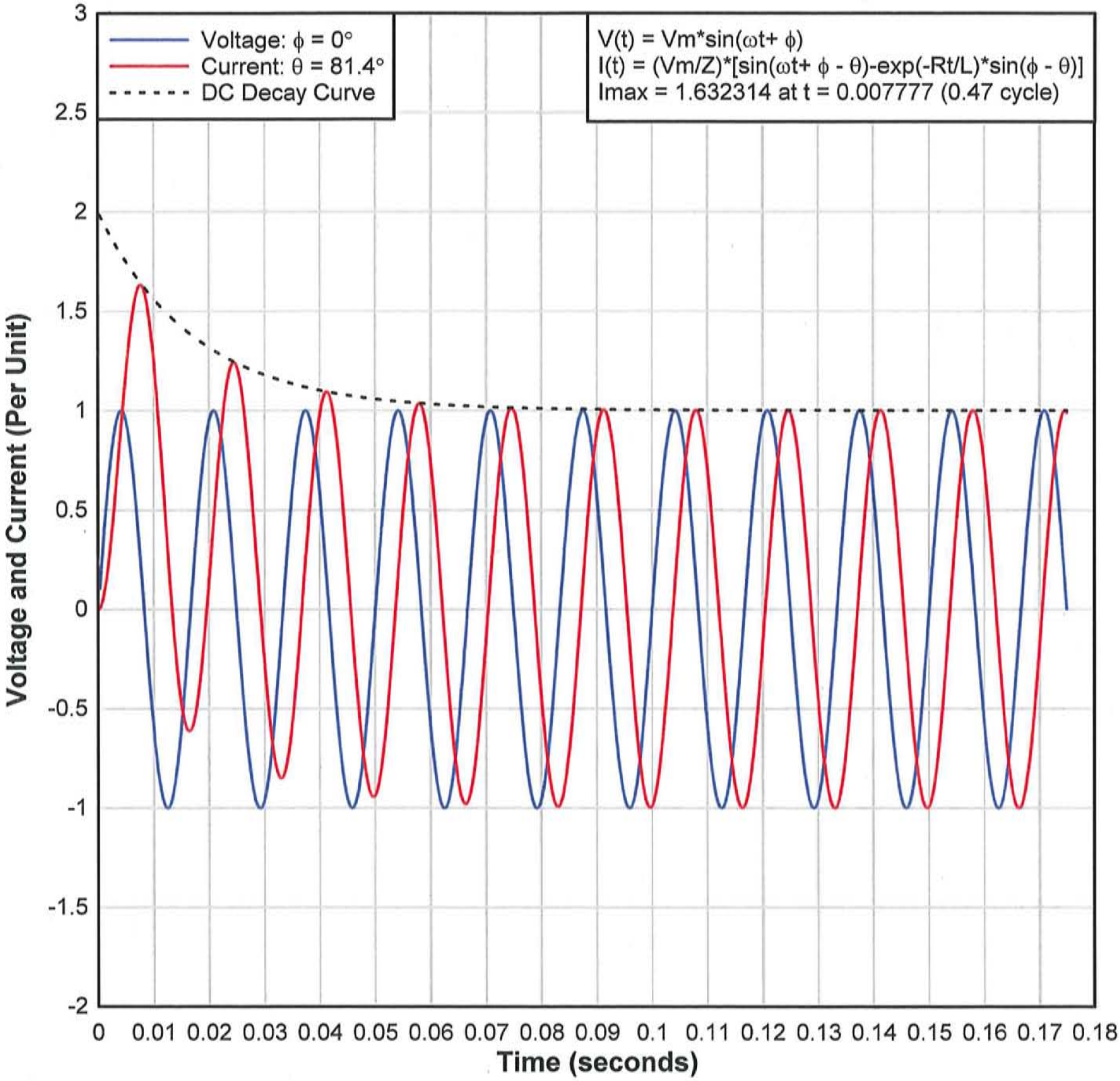
DC decay takes many, many cycles.
I peak at exactly 0.5 cycle.

Effect of X/R Ratio on Short Circuit Current
Voltage Closing Angle (Phase ϕ) = 0°
X/R = 25, PF = .04, θ = 87.7°

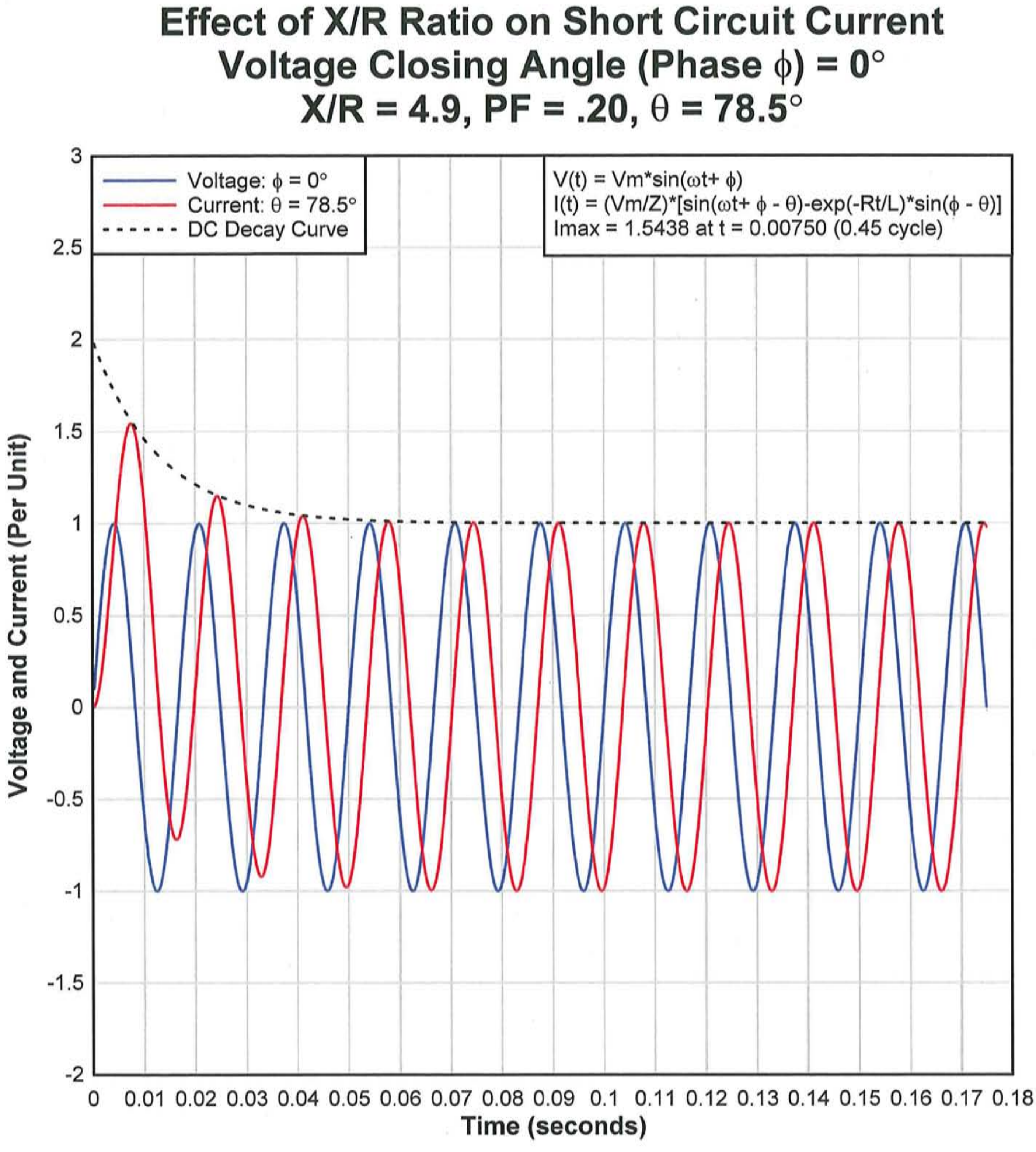


Notes: DC decay takes several cycles.
I peak at 0.48 cycle.

Effect of X/R Ratio on Short Circuit Current
Voltage Closing Angle (Phase ϕ) = 0°
X/R = 6.6, PF = .15, θ = 81.4°

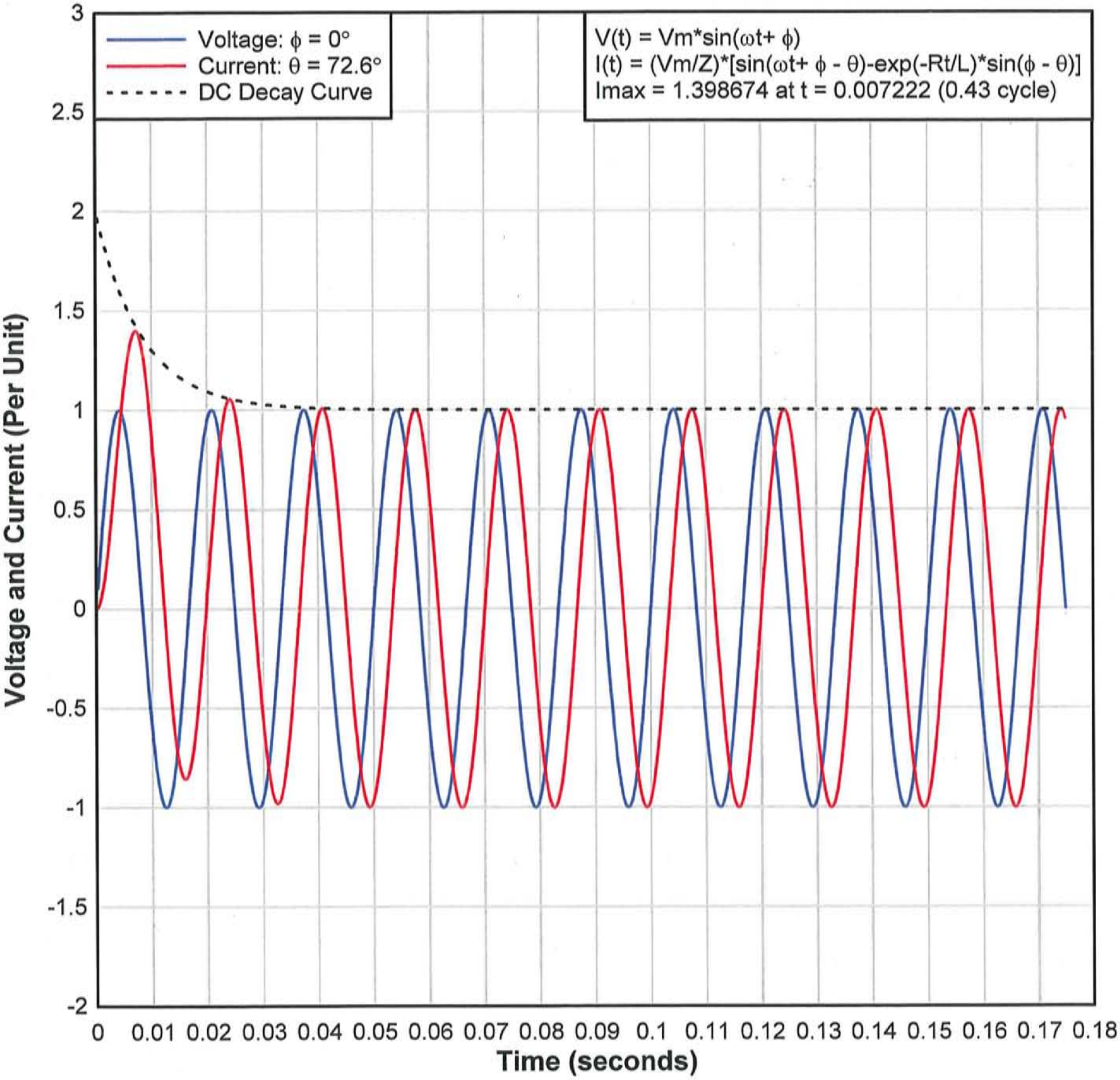


Notes: DC decay only takes a few cycles.
I peak at 0.47 cycle.

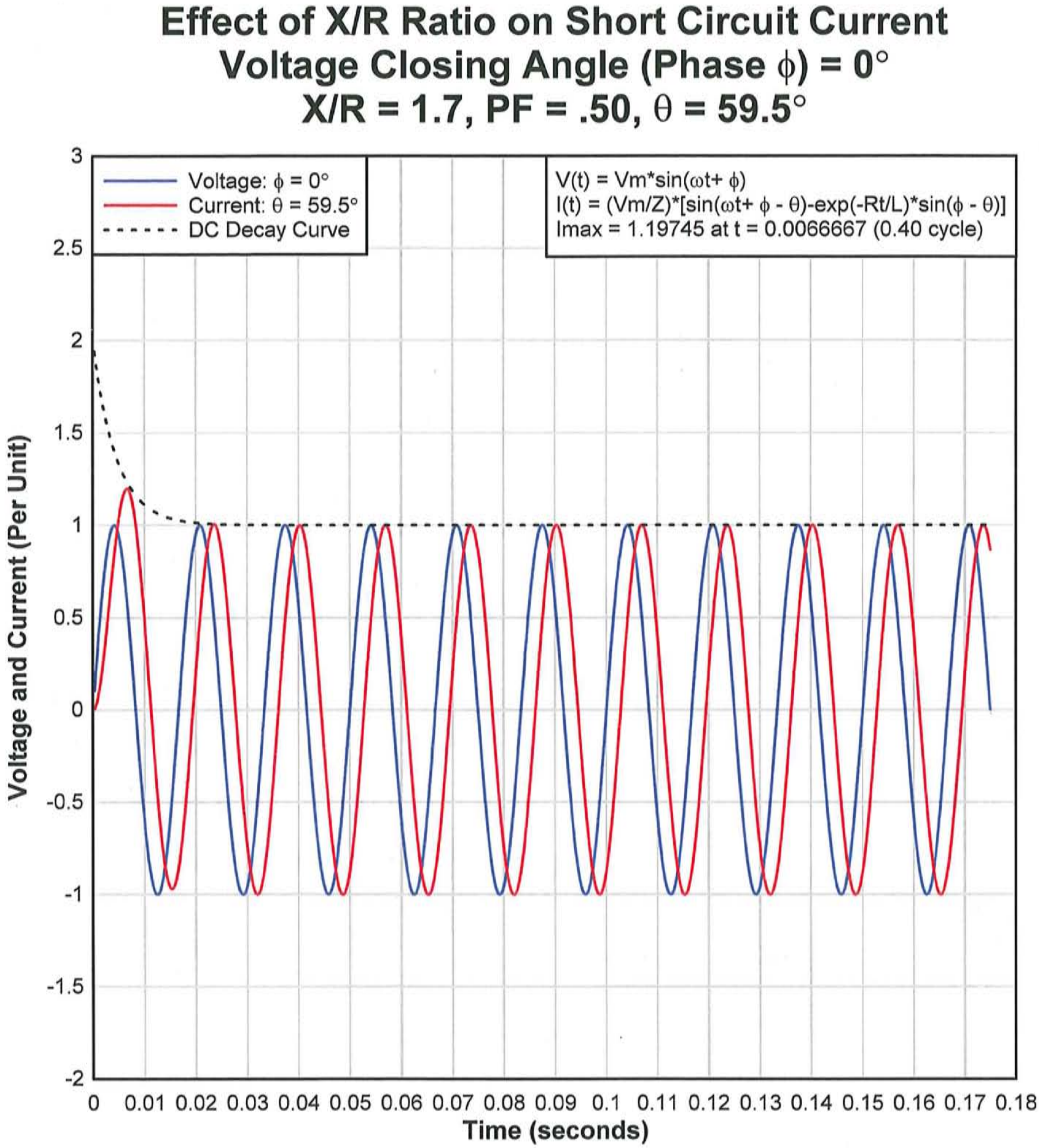


Notes: DC decay approx. 3 cycles.
I peak at 0.45 cycle.

Effect of X/R Ratio on Short Circuit Current
Voltage Closing Angle (Phase ϕ) = 0°
X/R = 3.2, PF = .30, θ = 72.6°



Notes: DC decay very quick.
I peak at 0.43 cycle.



Notes: DC decay in approx. 1 cycle.
I peak at 0.40 cycle.

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From analysis of the preceding plots:

1. Maximum DC offset occurs at voltage phase of 0° ($I_{max} = 2.0$ p.u.) or 180° ($I_{max} = -2.0$ p.u.); and θ the impedance angle is ≈ 90°. In other words, the impedance is essentially inductive.

2. If there is resistance in the circuit, then the exponential decay of the DC component comes into play.

3. When the Voltage Phase is set to 0° and the X/R ratios are varied, the following characteristics are noted:

   a) Very high X/R ratios (≥ 100), the time constant is long and decay of DC component takes many cycles.

   b) Very high X/R ratios - the peak current occurs nearly exactly one-half cycle into the fault. For lower X/R ratios, as the ratio decreases, the peak current occurs slightly earlier into the fault:

X/R = 100 I peak at t = 0.5 cycle
X/R = 25 I peak at t = 0.48 cycle
X/R = 6.6 I peak at t = 0.47 cycle
X/R = 4.9 I peak at t = 0.45 cycle
X/R = 3.2 I peak at t = 0.43 cycle
X/R = 1.7 I peak at t = 0.40 cycle

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(4) Maximum DC offset occurs at the zero crossing of the voltage wave form. At $\phi = 0°$ the voltage is zero and increasing; this results in maximum positive DC offset. At $\phi = 180°$ the voltage is zero and decreasing; this results in maximum negative DC offset.

(5) For typical X/R ratios seen by faults in a distribution system (e.g. $X/R < 10$) the DC component decays within a few cycles.

(6) EPRI research has shown that 60% of faults occur when voltage phase (closing angle) was within 5% of peak (angle 70°-90°) thus yielding currents with DC offsets far below the theoretical maximum of 2.0 p.u. Lightning faults are a different story. [See Fig. 7.5 in Electric Power Distribution Handbook by T. A. Short].

(7) The plots show that there are significant considerations involved when switching highly inductive loads or applying circuit breakers close to such loads such as in a substation with large transformers.

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Implications for Application of Circuit Breakers

Fault calculations are performed primarily in phasor (frequency) domain and it would be impractical to perform time domain transient analysis for every application.

It is desirable to have a simple multiplier to apply to the rms symmetrical fault currents calculated in the phasor domain in order to compensate for the DC offset, e.g. the asymmetrical current or total rms current.

To this end, the time domain equation previously derived can be simplified for a quasi-worst-case scenario as follows:

- Assume that the driving voltage for the fault always occurs when $\phi = 0$, the zero crossing when voltage is increasing.
- Assume that the argument $(\phi - \theta)$ of the sine functions always evaluate to $\pm \pi/2$ or $90°$ to yield a 1.0 maximum multiplier (we use $-\pi/2$ to produce a positive DC offset).
- Assume that the peak asymmetrical current occurs exactly one-half cycle into the fault.

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Modification of the equation:

$$I(t) = \frac{V_m}{|Z|}\left[\sin(\omega t + \phi - \theta) - e^{-(R/L)t} \cdot \sin(\phi - \theta)\right]$$

$$t = 0.5\,\text{cycle} = 0.5\left(\frac{1}{F}\right) = \frac{0.5}{F} = \frac{0.5}{\omega/2\pi} = \frac{(0.5)2\pi}{\omega} = \frac{\pi}{\omega}$$

$$(\phi - \theta) = -\pi/2$$

$$I_{max} = \frac{V_m}{|Z|}\left[\sin\left(\omega\left(\frac{\pi}{\omega}\right) - \frac{\pi}{2}\right) - e^{-(R/L)\left(\frac{\pi}{\omega}\right)} \cdot \sin(-\pi/2)\right]$$

$$I_{max} = \frac{V_m}{|Z|}\left[\underbrace{\sin\left(\frac{\pi}{2}\right)}_{1} - e^{-\frac{\pi}{(\omega L/R)}} \cdot \underbrace{\sin(-\pi/2)}_{-1}\right]$$

$$I_{max} = \frac{V_m}{|Z|}\left[1 + e^{-\frac{\pi}{(X/R)}}\right] \quad \text{where } X = \omega L$$

$$I_{max} = \frac{V_m}{|Z|} + \frac{V_m}{|Z|} e^{-\frac{\pi}{(X/R)}}$$

$$I_{max} = \underbrace{\frac{\sqrt{2}\,V_{rms}}{|Z|}}_{AC} + \underbrace{\frac{\sqrt{2}\,V_{rms}}{|Z|} e^{-\frac{\pi}{(X/R)}}}_{DC}$$

$$I_{max} = \underbrace{\sqrt{2}\,I_{rms}}_{\substack{\text{AC rms sym}\\ \text{current}\\ \text{steady-state}}} + \underbrace{\sqrt{2}\,I_{rms} \cdot e^{-\frac{\pi}{(X/R)}}}_{\substack{\text{max DC offset}\\ \text{transient}}}$$

In this manner, simple adjustment for the transient asymmetry of the fault can be made using the rms sym fault current (phasor domain) and the system X/R ratio.

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$$I_{peak} = \sqrt{2} I_{rms} \left[1 + e^{-\frac{\pi}{(X/R)}}\right]$$

This is the most industry-accepted approximation that is used, but it gives an approximation that is slightly low [Elec. Power Dist. Handbook].

As used in ANSI/IEEE C37.13 (Low Voltage AC Power Circuit Breakers) for unfused breakers rated based on peak current:

Unfused breakers are rated and tested down to power factor of 15% (up to X/R ratio of 6.6).

Obtain a multiplying factor (MF) to apply to rms currents with X/R ratios higher than 6.6 (e.g. normalize equation to X/R = 6.6).

$$MF = \frac{I_{peak\ X/R=higher}}{I_{peak\ X/R=6.6}} = \frac{\sqrt{2} \cancel{I_{rms}} \left[1 + e^{-\frac{\pi}{(X/R)}}\right]}{\sqrt{2} \cancel{I_{rms}} \left[1 + e^{-\pi/6.6}\right]} = \frac{\sqrt{2}\left[1 + e^{-\frac{\pi}{(X/R)}}\right]}{2.29}$$

Use this equation to recreate Table 3 in ANSI/IEEE C37.13 for unfused circuit breakers:

System Short-Circuit Power Factor %	System X/R Ratio	Multiplying Factor for Calculated rms sym fault current
15	6.6	1.00
12	8.27	1.04
10	9.95	1.07
8.5	11.72	1.09
7	14.25	1.11
5	20.0	1.14

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As used in ANSI/IEEE C37.13 (Low Voltage Power Circuit Breakers) for fused circuit breakers rated based on total rms current (asymmetrical):

IEEE Definition from C37.010,

$$\text{Total rms current} = \sqrt{(I_{sym})^2 + (I_{DC\ component})^2}$$

$$I_{\substack{total \\ rms\ asym}} = \sqrt{(I_{rms})^2 + \left(\sqrt{2}\, I_{rms}\, e^{-\frac{\pi}{(X/R)}}\right)^2} \longleftarrow \text{DC component from simplified equation}$$

$$I_{total} = \sqrt{I_{rms}^2 + 2 I_{rms}^2 e^{-\frac{2\pi}{(X/R)}}}$$

$$I_{total} = \sqrt{I_{rms}^2 \left(1 + 2e^{-\frac{2\pi}{(X/R)}}\right)}$$

$$I_{total} = \sqrt{I_{rms}^2} \cdot \sqrt{1 + 2e^{-\frac{2\pi}{(X/R)}}}$$

$$I_{\substack{total \\ rms\ asym}} = I_{rms} \cdot \sqrt{1 + 2e^{-\frac{2\pi}{(X/R)}}}$$

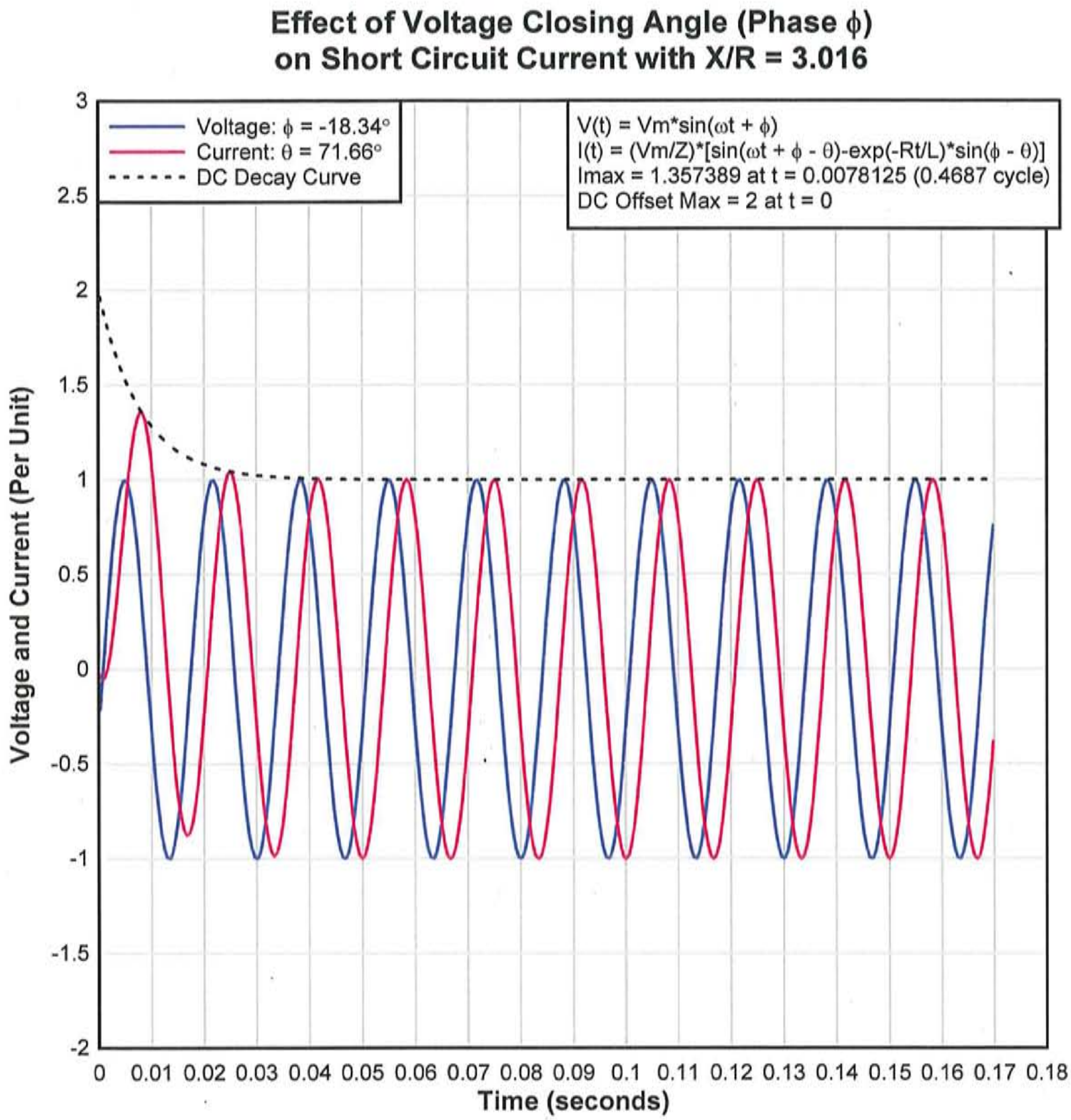
Fused breakers are rated and tested down to power factor of 20% (up to X/R ratio of 4.9).

Obtain a multiplying factor (MF) to apply to rms sym currents with X/R ratios higher than 4.9 (e.g. normalize equation to X/R = 4.9).

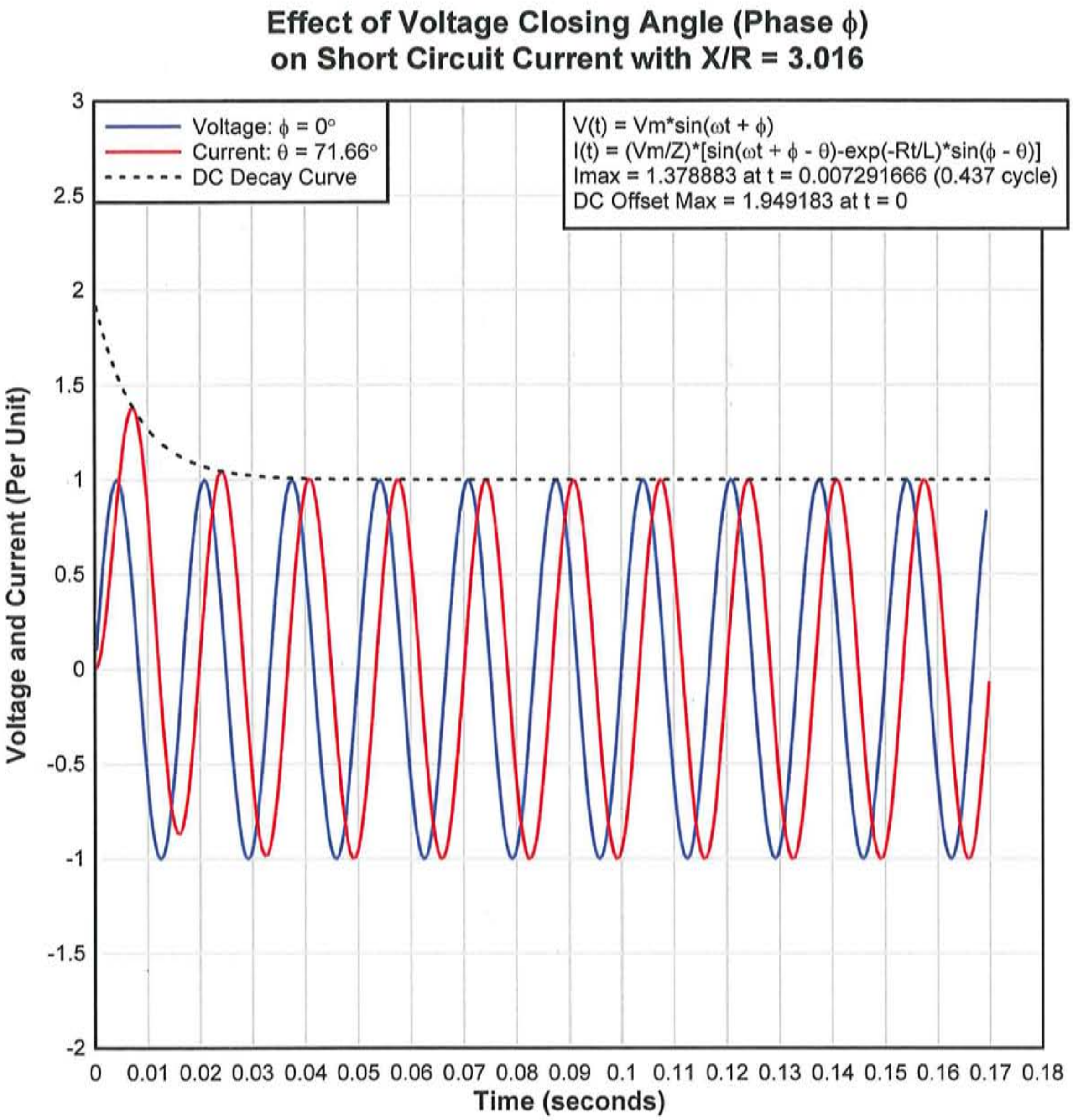
$$MF = \frac{I_{\substack{total \\ X/R = higher}}}{I_{\substack{total \\ X/R = 4.9}}} = \frac{\cancel{I_{rms}}\sqrt{1+2e^{-\frac{2\pi}{(X/R)}}}}{\cancel{I_{rms}}\sqrt{1+2e^{-\frac{2\pi}{4.9}}}} = \frac{\sqrt{1+2e^{-\frac{2\pi}{(X/R)}}}}{1.25}$$

Use this equation to recreate Table 3 in ANSI/IEEE C37.13 for fused circuit breakers:

System Short-Circuit Power Factor %	System X/R Ratio	Multiplying Factor for Calculated rms sym fault current
20	4.9	1.00
15	6.6	1.07
12	8.27	1.12
10	9.95	1.15
8.5	11.72	1.18
7	14.25	1.21
5	20.0	1.26



Notes: When $(\phi - \theta)$ is $90°$ and ϕ is not $0°$, maximum DC offset of 2.0 pu is produced, but does <u>not</u> result in maximum possible value of current.



Notes: Maximum value of current is only produced when $(\phi - \theta)$ is $\pm 90°$ <u>and</u> $\phi = 0°$ or $\phi = 180°$

Method 3: Solve First-Order Equation

Using general solution method for linear differential equations with constant coefficients. General solution of an inhomogeneous equation is the sum of solutions to homogeneous equation and solutions to particular integral, combined with solution to initial value problem.

Inhomogeneous equation: $\frac{dI}{dt} + \frac{R}{L} I(t) = \frac{V_m}{L} \sin(\omega t + \phi)$

Solution to Homogeneous Equation:

Homogeneous equation: $\frac{dI}{dt} + \frac{R}{L} I(t) = 0$

characteristic polynomial: $(r + \frac{R}{L}) = 0 \quad r = -\frac{R}{L}$
single (simple) root with multiplicity 1

The general solution (complimentary function) is of the form,

$$I(t)_h = C_1 e^{-(\frac{R}{L})t}$$

(cont.)

Solution to Particular Integral By the Method of Undetermined Coefficients

For inhomogeneous equation, $\frac{dI}{dt} + \frac{R}{L}I(t) = \frac{V_m}{L}\sin(\omega t+\phi)$

solution to particular integral is of the form

$$I(t)_p = C\cos(\omega t+\phi) + D\sin(\omega t+\phi)$$

since $\cos(\omega t+\phi)$ and/or $\sin(\omega t+\phi)$ are not contained in the complementary function (they are not solutions to the homogeneous equation).

To determine coefficients C and D, substitute $I(t)_p$ into original inhomogeneous equation:

$$\frac{dI(t)_p}{dt} + \frac{R}{L}I(t)_p = \frac{V_m}{L}\sin(\omega t+\phi)$$

$$-\omega C\sin(\omega t+\phi) + \omega D\cos(\omega t+\phi) + \frac{R}{L}C\cos(\omega t+\phi) + \frac{R}{L}D\sin(\omega t+\phi) = \frac{V_m}{L}\sin(\omega t+\phi)$$

Equate coefficients of corresponding terms on either side of the equation:

Coefficients of $\sin(\omega t+\phi)$: $\quad -\omega C + \frac{R}{L}D = \frac{V_m}{L}$

Coefficients of $\cos(\omega t+\phi)$: $\quad \omega D + \frac{R}{L}C = 0$

Solve for coefficients,

$$\omega D = -\frac{R}{L}C \Rightarrow D = \frac{-R}{\omega L}C$$

$$-\omega C + \frac{R}{L}\left(\frac{-R}{\omega L}C\right) = \frac{V_m}{L}$$

(cont.)

$$-\omega C - \frac{R^2}{\omega L^2} C = \frac{V_m}{L} \Rightarrow C = \frac{V_m}{L}\left(\frac{1}{-\omega - \frac{R^2}{\omega L^2}}\right) = \frac{-V_m \cdot \omega L^2}{L(R^2+\omega^2 L^2)}$$

$$C = \frac{-V_m \cdot \omega L}{R^2+(\omega L)^2} \qquad D = \frac{-R}{\omega L}\left(\frac{-V_m \cdot \omega L}{R^2+(\omega L)^2}\right) = \frac{V_m \cdot R}{R^2+(\omega L)^2}$$

Then, $I(t)_p = \frac{-V_m \cdot \omega L}{R^2+(\omega L)^2}\cos(\omega t+\phi) + \frac{V_m \cdot R}{R^2+(\omega L)^2}\sin(\omega t+\phi)$

and general solution to inhomogeneous equation is of the form,

$$I(t) = I(t)_h + I(t)_p$$

$$I(t) = C_1 e^{-\left(\frac{R}{L}\right)t} - \frac{V_m \cdot \omega L}{R^2+(\omega L)^2}\cos(\omega t+\phi) + \frac{V_m \cdot R}{R^2+(\omega L)^2}\sin(\omega t+\phi)$$

Find solution to Initial Value Problem

Assume that at $t=0$, $I(0)=0$ so that,

$$0 = C_1 - \frac{V_m \cdot \omega L}{R^2+(\omega L)^2}\cos\phi + \frac{V_m \cdot R}{R^2+(\omega L)^2}\sin\phi$$

$$C_1 = \frac{V_m}{R^2+(\omega L)^2}(\omega L\cos\phi - R\sin\phi)$$

The (complete) general solution is now,

$$I(t) = \frac{V_m}{R^2+(\omega L)^2}(\omega L\cos\phi - R\sin\phi)e^{-\left(\frac{R}{L}\right)t} - \frac{V_m}{R^2+(\omega L)^2}\left[\omega L\cos(\omega t+\phi) - R\sin(\omega t+\phi)\right]$$

Rearranging and simplifying:

$$R^2+(\omega L)^2 = |Z|^2$$

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$$I(t) = \frac{V_m}{|Z|^2}\left[(\omega L\cos\phi - R\sin\phi)\cdot e^{-\left(\frac{R}{L}\right)t} - \left[\omega L\cos(\omega t+\phi) - R\sin(\omega t+\phi)\right]\right]$$

$$I(t) = \frac{V_m}{|Z|^2}\left[R\sin(\omega t+\phi) - \omega L\cos(\omega t+\phi) - e^{-\left(\frac{R}{L}\right)t}(R\sin\phi - \omega L\cos\phi)\right]$$

Apply trigonometric identity,

$$a\sin x - b\cos x = \pm\sqrt{a^2+b^2}\cdot\sin\left(x - \tan^{-1}\frac{b}{a}\right), \quad \theta = \tan^{-1}\frac{b}{a}$$

$$I(t) = \frac{V_m}{|Z|^2}\left[\pm\sqrt{R^2+(\omega L)^2}\sin(\omega t+\phi-\theta) - e^{-\left(\frac{R}{L}\right)t}\cdot\pm\sqrt{R^2+(\omega L)^2}\sin(\phi-\theta)\right]$$

$$\sqrt{R^2+(\omega L)^2} = |Z|$$

$$I(t) = \frac{V_m\cdot|Z|}{|Z|^2}\cdot\left[\sin(\omega t+\phi-\theta) - e^{-\left(\frac{R}{L}\right)t}\sin(\phi-\theta)\right]$$

Finally,

$$I(t) = \frac{V_m}{|Z|}\left[\sin(\omega t+\phi-\theta) - e^{-\left(\frac{R}{L}\right)t}\cdot\sin(\phi-\theta)\right] \checkmark$$

Method 4: Solve first-order equation by Transient and Steady-State components. Use phasor method for steady-state component with AC source.

$$\frac{dI}{dt} + \frac{R}{L} I(t) = \frac{V_m}{L} \sin(\omega t + \phi)$$

From property of linearity,

$$\frac{dI}{dt} + \frac{R}{L} I(t) = 0 + \frac{V_m}{L} \sin(\omega t + \phi)$$

Transient Solution (Natural Response):

$$\frac{dI}{dt} + \frac{R}{L} I(t) = 0$$

Steady-State Solution (Forced Response):

$$\frac{dI}{dt} + \frac{R}{L} I(t) = \frac{V_m}{L} \sin(\omega t + \phi)$$

$$\underbrace{I(t)}_{\text{Total solution}} = \underbrace{I(t)_{TR}}_{\text{Transient solution}} + \underbrace{I(t)_{SS}}_{\text{steady-state solution}}$$

Note: This form does not yet incorporate the specific solution to initial value problem

For Transient Component:

$$\frac{dI}{dt} + \frac{R}{L} I(t) = 0$$

characteristic polynomial: $r + \frac{R}{L} = 0 \quad r = \frac{-R}{L}$

$$I(t)_{TR} = C_1 e^{-(R/L)t}$$

single (simple) root
multiplicity 1

For Steady-State Component!

since forcing function (AC source) is a sinusoid, the phasor method can be used:

$$\frac{dI}{dt} + \frac{R}{L} I(t)_{ss} = \frac{V_m}{L} \sin(\omega t + \phi)$$

Express in terms of cosine (adopt a cosine reference):

$$\frac{dI}{dt} + \frac{R}{L} I(t)_{ss} = \frac{V_m}{L} \cos(\omega t + \phi - \tfrac{\pi}{2})$$

Express with phasor equivalents:

$$\frac{d}{dt}\tilde{I}\cancel{e^{j\omega t}} + \frac{R}{L}\tilde{I}\cancel{e^{j\omega t}} = \frac{\tilde{V}_m}{L}\cancel{e^{j\omega t}}, \quad \tilde{V}_m = V_m e^{j\phi} \cdot e^{-j\frac{\pi}{2}}$$

$$j\omega \tilde{I} + \frac{R}{L}\tilde{I} = \frac{\tilde{V}_m}{L} \Rightarrow j\omega L \tilde{I} + R\tilde{I} = \tilde{V}_m$$

$$\tilde{I}(R + j\omega L) = \tilde{V}_m \Rightarrow \tilde{I} = \frac{\tilde{V}_m}{R + j\omega L}$$

$$R + j\omega L = \tilde{Z} \Rightarrow \tilde{I} = \frac{\tilde{V}_m}{\tilde{Z}} = \frac{V_m \angle \phi - \frac{\pi}{2}}{Z \angle \theta} \qquad \theta = \tan \frac{\omega L}{R}$$

$$\tilde{I} = \frac{V_m}{|Z|} \angle \phi - \theta - \frac{\pi}{2}$$

(cont.)

Convert back to time domain:

$$\tilde{I}e^{j\omega t} = Re\left\{\frac{V_m}{|Z|}e^{j(\phi-\theta-\frac{\pi}{2})}e^{j\omega t}\right\}$$

$$I(t)_{ss} = \frac{V_m}{|Z|}\cos\left(\omega t+\phi-\theta-\frac{\pi}{2}\right)$$

or $$I(t)_{ss} = \frac{V_m}{|Z|}\sin(\omega t+\phi-\theta)$$

Total solution is now,

$$I(t) = C_1 e^{-(\frac{R}{L})t} + \frac{V_m}{|Z|}\sin(\omega t+\phi-\theta)$$

Now must solve for initial conditions.
Note that at $t=0^+$

$$I(0^+) = C_1 + I_{ss}(0^+) \Rightarrow C_1 = I(0^+) - I_{ss}(0^+)$$

General form of total solution is given by,

$$I(t) = \left[I(0^+) - I_{ss}(0^+)\right]e^{-(R/L)t} + I(t)_{ss}$$

In this case, $I(0^+)=0$ at $t=0^+$ so that,

$$0 = C_1 + \frac{V_m}{|Z|}\sin(\phi-\theta) \Rightarrow C_1 = -\frac{V_m}{|Z|}\sin(\phi-\theta)$$

Then,

$$I(t) = -\frac{V_m}{|Z|}\sin(\phi-\theta)\cdot e^{-(\frac{R}{L})t} + \frac{V_m}{|Z|}\sin(\omega t+\phi-\theta)$$

or

$$I(t) = \frac{V_m}{|Z|}\left[\sin(\omega t+\phi-\theta) - e^{-(\frac{R}{L})t}\cdot\sin(\phi-\theta)\right] \checkmark$$